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Department of Electrical Engineering and Computer Science

6.685 Electric Machines　電機機械

課堂講義 1：電磁力　　2005 年 9 月 5 日

Class Notes 1: Electromagnetic Forces

　September 5, 2005

1 導論

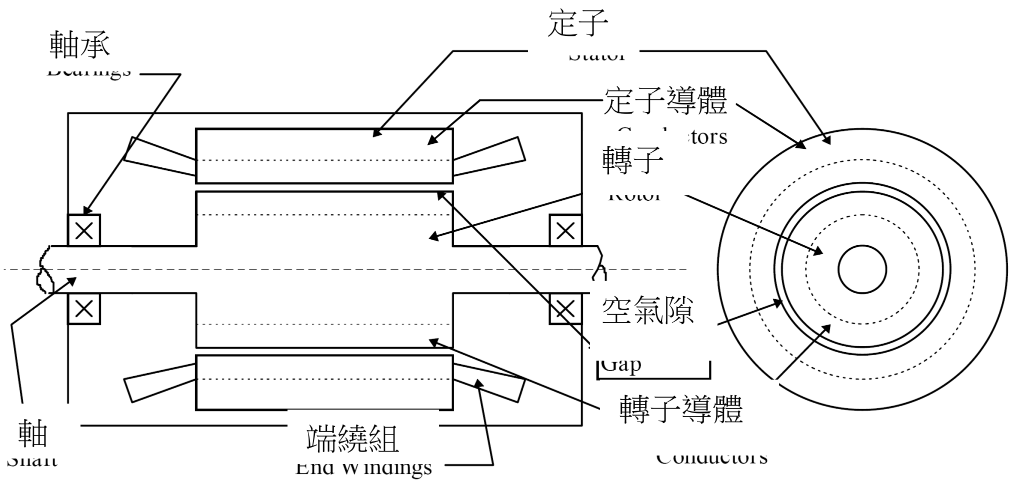

圖 1：電機機械形式

1 Introduction

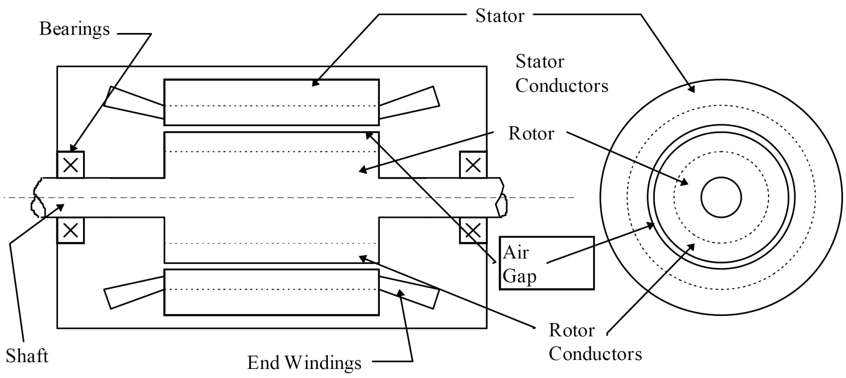

Figure 1: Form of Electric Machine

本章討論一些電機機械包含的基本程序。在能量轉換程序節將討論兩種估計電磁力的主要方法：其一是依據熱力學理論（能量守恆[1]），另一個則為磁場方法（馬克斯威爾應力張量[2]）。首先將以概念性的旋轉電機介紹這個主題。
This section of notes discusses some of the fundamental processes involved in electric machinery. In the section on energy conversion processes we examine the two major ways of estimating electromagnetic forces: those involving thermodynamic arguments (conservation of energy) and field methods (Maxwell's Stress Tensor). But first it is appropriate to introduce the topic by describing a notional rotating electric machine.

電機機械有許多種不同型式，其容量範圍又極為廣泛，小焉者用來引起行動電話或呼叫器震動（都是旋轉電機）；大焉者如渦輪發電機，其容量達數十億瓦[3]。大部分我們熟悉的機械是旋轉機，但是線性馬達也廣被使用，從織布機的飛梭到搬運設備以及遊樂園的載具。目前正在發展的是用來發射飛機的大型線性感應機。本課程的目標是發展一套分析的基礎，以解釋種種不同機器的運轉原理。我們將從最普通的電機機械構造圖開始。
Electric machinery comes in many different types and a strikingly broad range of sizes, from those little machines that cause cell 'phones and pagers to vibrate (yes, those are rotating electric machines) to turbine generators with ratings upwards of a Gigawatt. Most of the machines with which we are familiar are rotating, but linear electric motors are widely used, from shuttle drives in weaving machines to equipment handling and amusement park rides. Currently under development are large linear induction machines to be used to launch aircraft. It is our purpose in this subject to develop an analytical basis for understanding how all of these different machines work. We start, however, with a picture of perhaps the most common of electric machines.

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[1] 能量守恆 conservation of energy
[2] 馬克斯威爾應力張量 Maxwell's Stress Tensor
[3] 十億瓦 Gigawatt

2. 電機機械說明

圖 1 是傳統感應馬達的示意圖，這是很普通的電機機械，可以作為參考基準，大部分其他電機機械的運轉方式與此相同，即使有不同的運轉方式，也可以參考感應機。

2 Electric Machine Description:

Figure 1 is a cartoon drawing of a conventional induction motor. This is a very common type of electric machine and will serve as a reference point. Most other electric machines operate in a fashion which is the same as the induction machine or which differ in ways which are easy to reference to the induction machine.

我們要學習的大部分機器（但不是全部！）是屬於這種型態，機器的轉子裝於一根軸[4]上，這根軸由軸承[5]支撐，通常（但不是百分之百）轉子[6]裝在裡面。雖然圖上畫的轉子是圓形，但不表示所有轉子都必然是這種形狀。圖上也示出轉子的導線，但有時轉子為內藏或外裝的永久磁鐵[7]，有時只是一枚奇形怪狀的鋼件（例如可變磁阻電機[8]）。圖上所示的定子[9]裝在外面，並且有繞組[10]。我們將要討論的機器，大部分定子繞組就是電樞[11]，也就是輸入電功率[12]的元件（在直流[13]或泛用馬達[14]，此情況反過來，電樞位於轉子，後面將會討論到）。

Most (but not all!) machines we will be studying have essentially this morphology. The rotor of the machine is mounted on a shaft which is supported on some sort of bearing(s). Usually, but not always, the rotor is inside. I have drawn a rotor which is round, but this does not need to be the case. I have also indicated rotor conductors, but sometimes the rotor has permanent magnets either fastened to it or inside, and sometimes (as in Variable Reluctance Machines) it is just an oddly shaped piece of steel. The stator is, in this drawing, on the outside and has windings. With most of the machines we will be dealing with, the stator winding is the armature, or electrical power input element.(In DC and Universal motors this is reversed, with the armature contained on the rotor: we will deal with these later).

大部分的電機機械，轉子和定子都用高導磁性材料（鋼或磁性鐵）製造，許多常用的機器例如感應馬達[15]，轉子和定子是用很薄的矽鋼片疊成，上面開槽[16]以安裝轉子

[4] 軸 shaft
[5] 軸承 bearing
[6] 轉子 rotor
[7] 永久磁鐵 permanent magnet
[8] 可變磁阻電機 Variable Reluctance Machine
[9] 定子 stator
[10] 繞組 windings
[11] 電樞 armature
[12] 電功率 electrical power
[13] 直流 direct current (DC)
[14] 泛用馬達 Universal motor
[15] 感應馬達 induction motor
[16] 槽 slot

和定子的導線[17]。

In most electrical machines the rotor and the stator are made of highly magnetically permeable materials: steel or magnetic iron. In many common machines such as induction motors the rotor and stator are both made up of thin sheets of silicon steel. Punched into those sheets are slots which contain the rotor and stator conductors.

圖 2 表示感應機的一部份，但是稍作變形而拉直氣隙[18]，就像此機器的半徑無窮大。這是一種很方便的畫法，而且可以引導出很有用的分析方法。

Figure 2 is a picture of part of an induction machine distorted so that the air-gap is straightened out (as if the machine had infinite radius).This is actually a convenient way of drawing the machine and, we will find, leads to useful methods of analysis.

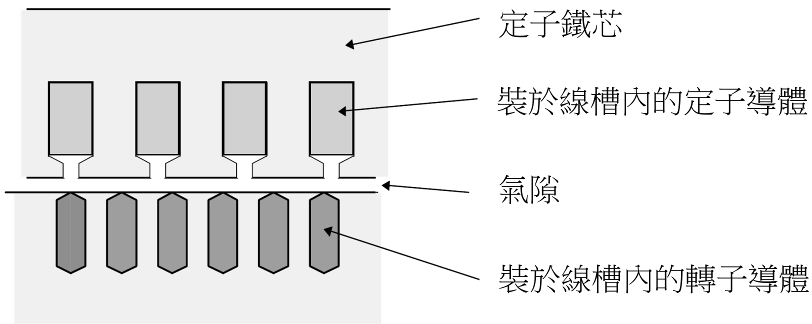

圖 2：裝於線槽內之繞組

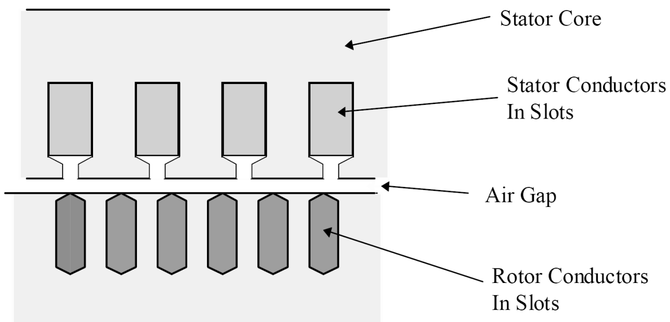

Figure 2: Windings in Slots

請注意此機器的氣隙大小 g 遠小於半徑 r，氣隙長度為 ℓ，此機器能夠旋轉乃是在氣隙產生剪應力（同時產生另一個「反電壓[19]」的副作用）。可以算出平均氣隙剪應力[20] τ。機器產生的總轉矩[21]為轉子表面的力乘以力矩[22]（即轉子半徑）。

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[17] 導線 conductor
[18] 氣隙 air-gap
[19] 反電壓 back voltage
[20] 剪應力 shear stress
[21] 轉矩 torque
[22] 力矩 moment

What is important to note for now is that the machine has an air gap *g* which is relatively small (that is, the gap dimension is much less than the machine radius *r*). The air-gap also has a physical length ℓ. The electric machine works by producing a shear stress in the air-gap (with of course side effects such as production of "back voltage"). It is possible to define the average air-gap shear stress, which we will refer to as *τ*. Total developed torque is force over the surface area times moment (which is rotor radius):

$$T = 2\pi r^2 \ell < \tau >$$

此裝置傳送的功率為轉矩乘以轉速，等於力乘以表面速率，因表面速率[23]為 $u= r\Omega$，
Power transferred by this device is just torque times speed, which is the same as force times surface velocity, since surface velocity is $u= r\Omega$:

$$P_m = \Omega T = 2\pi r \ell < \tau > u$$

「動轉子體積」為轉矩對體積的比值：
If we note that active rotor volume is the ratio of torque to volume is just:

$$\frac{T}{V_r} = 2 < \tau >$$

有關此機器容積的討論有兩種：首先，上面所計算的體積並非機器的全部體積，因為定子並未計算在內。欲根據轉子體積來估計整部機器的體積，其方法非常複雜而且瑣碎，留待以後再討論。其次，需要估算有效的平均剪應力。假設徑向[24]的磁通[25]密度 B_r 和定子表面電流密度 K_z 都是正弦波[26]，其形式如下：
Now, determining what can be done in a volume of machine involves two things. First, it is clear that the volume we have calculated here is not the whole machine volume, since it does not include the stator. The actual estimate of total machine volume from the rotor volume is actually quite complex and detailed and we will leave that one for later. Second, we need to estimate the value of the useful average shear stress. Suppose both the radial flux density B_r and the stator surface current density K_z are sinusoidal flux waves of the form:

$$B_r = \sqrt{2} B_0 \cos(p\theta - \omega t)$$

$$K_z = \sqrt{2} K_0 \cos(p\theta - \omega t)$$

此處假設這兩個量為同相[27]位，其方向正好可以產生理想的轉矩，而能得到「最佳」的結合。則平均的表面牽引力[28]為：
Note that this assumes these two quantities are exactly in phase, or oriented to ideally produce torque, so we are going to get an "optimistic" bound here. Then the average

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[23] 表面速率 surface velocity
[24] 徑向 radial
[25] 磁通 flux
[26] 正弦波 sinusoidal wave
[27] 相 phase
[28] 表面牽引力 surface traction

value of surface traction is:

$$<\tau>=\frac{1}{2\pi}\int_0^{2\pi} B_r K_z d\theta = B_0 K_0$$

機器內的磁通密度受到所使用的磁性材料（鐵）性質的限制，而電流密度則由絕緣材料的耐溫極限和冷卻能力決定之。實用上，電機機械所產生的剪應力大小範圍不寬，從小機器的數 kPa 到大型而冷卻良好的機器的 100kPa 左右。

The magnetic flux density that can be developed is limited by the characteristics of the magnetic materials (iron) used. Current densities are a function of technology and are typically limited by how much effort can be put into cooling and the temperature limits of insulating materials. In practice, the range of shear stress encountered in electric machinery technology is not terribly broad: ranging from a few kPa in smaller machines to about 100 kPa in very large, well cooled machines.

一般而言，電機機械是產生轉矩的設備，亦即其容量受到剪應力作用和機器尺寸所決定。由於功率是轉矩乘以轉速，高功率密度機器必然需有高轉速。當然轉速也因為離心力的問題而受到限制。

It is usually said that electric machines are torque producing devices, meaning tht they are defined by this shear stress mechanism and by physical dimensions. Since power is torque times rotational speed, high power density machines necessarily will have high shaft speeds. Of course there are limits on rotational speed as well, arising from centrifugal forces which limit tip velocity.

我們將從了解產生電磁力的作用開始，來了解電機機械的運轉原理。

Our first step in understanding how electric machinery works is to understand the mechanisms which produce forces of electromagnetic origin.

3. 能量轉換程序

電動機內的能量轉換程序可以想成如下簡單的敘述：在「穩態[29]」中，輸入到電動機的電功率就是每相端子[30]所輸入的電功率和，

3 Energy Conversion Process:

In a motor the energy conversion process can be thought of in simple terms. In “steady state”, electric power input to the machine is just the sum of electric power inputs to the different phase terminals:

$$P_e = \sum_i v_i i_i$$

機械功率為轉矩乘以轉速

Mechanical power is torque times speed:

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[29] 穩態 steady state

[30] 端子 terminal

$$P_m = T\Omega$$

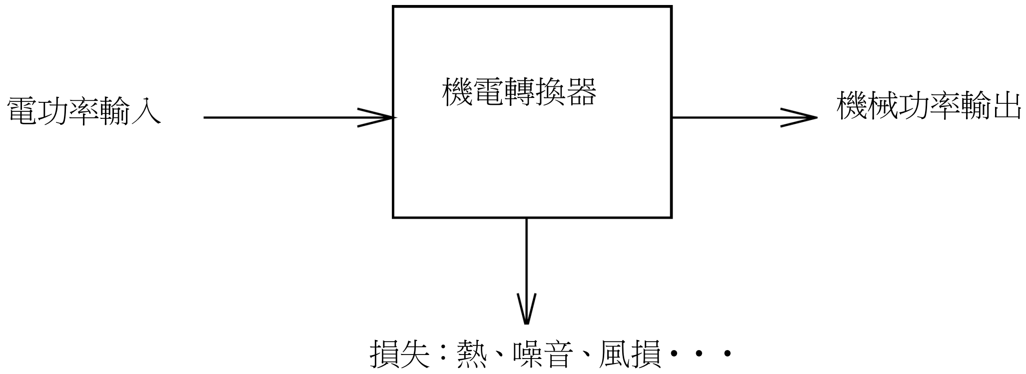

圖 3 能量轉換程序

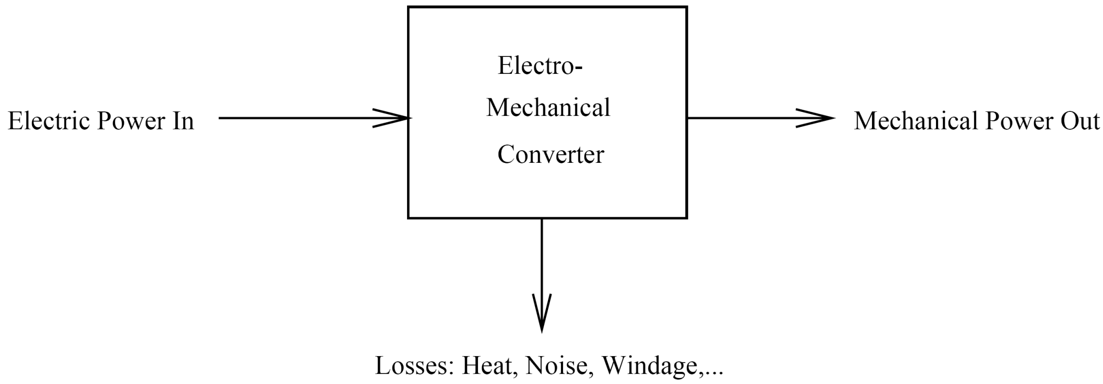

Figure 3: Energy Conversion Process

兩者之差即為總損失：
And the sum of the losses is the difference:

$$P_d = P_e - P_m$$

由於大部分電機機械的損失很小，有時可以將機械功率當作等於電功率。事實上許多情況下損失是被忽略的，在「熱力學」討論力量密度時，常常利用這一優點，將電機機械當做「守恆」或無損失的能量轉換系統。
It will sometimes be convenient to employ the fact that, in most machines, dissipation is small enough to approximate mechanical power with electrical power. In fact, there are many situations in which the loss mechanism is known well enough that it can be idealized away. The “thermodynamic” arguments for force density take advantage of this and employ a “conservative” or lossless energy conversion system.

3.1 用能量方法來推導電磁力

3.1 Energy Approach to Electromagnetic Forces:

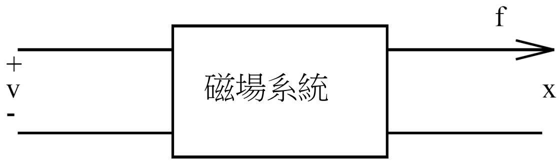

圖 4：守恆的磁場系統

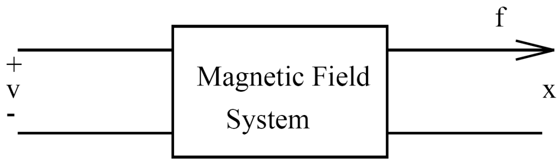

Figure 4: Conservative Magnetic Field System

圖 4 所示的機電系統有兩組端子：電的端子和機械端子。若系統的磁場儲存著能量，所儲存的能量大小乃根據系統的狀態。在此例中，是由下述三個變數中的兩個所決定：磁通(λ)（譯註：本講義常常將磁通(φ)和磁交鏈(λ)都稱為磁通 flux，翻譯時將忠於原著。）、電流(i)和機械位置(x)。事實上只要稍加思考，就可以發現此一狀態是雙變數單值函數，而所儲存的能量與系統如何變成這種狀態的過程無關。

To start, consider some electromechanical system which has two sets of “terminals”, electrical and mechanical, as shown in Figure 4. If the system stores energy in magnetic fields, the energy stored depends on the state of the system, defined by (in this case)two of the identifiable variables: flux (λ), current (i) and mechanical position (x). In fact, with only a little reflection, you should be able to convince yourself that this state is a single-valued function of two variables and that the energy stored is independent of how the system was brought to this state.

所有機電轉換裝置都有損失，所以並非能量守恆。但是磁場系統產生力，基本上是守恆的，因為其狀態和所儲存的能量可以只用兩個變數描述。系統的「歷史」並不重要。

Now, all electromechanical converters have loss mechanisms and so are not themselves conservative. However, the magnetic field system that produces force is, in principle, conservative in the sense that its state and stored energy can be described by only two variables. The “history” of the system is not important.

可以依下述方法來選擇變數，輸入到這守恆系統的電功率為：

It is possible to choose the variables in such a way that electrical power into this conservative system is:

$$P^e = vi = i\frac{d\lambda}{dt}$$

相同的，系統輸出的機械功率為：

Similarly, mechanical power out of the system is:

$$P^m = f^e \frac{dx}{dt}$$

兩者之差即是系統所儲存能量的變化率：
The difference between these two is the rate of change of energy stored in the system:

$$\frac{dW_m}{dt} = P^e - P^m$$

接著可以計算讓此系統由一個狀態改變到另一個狀態所需要的能量變化：
It is then possible to compute the change in energy required to take the system from one state to another by:

$$W_m(a) - W_m(b) = \int_b^a i d\lambda - f^e dx$$

這系統的兩個狀態可以用 $a = (\lambda_a, x_a)$ 和 $b = (\lambda_b, x_b)$代表之。
若儲存於此系統的能量使用狀態變數λ和 x 來描述，儲存能量的微分為：
where the two states of the system are described by $a = (\lambda_a, x_a)$ and $b = (\lambda_b, x_b)$
If the energy stored in the system is described by two state variables, λ and x, the total differential of stored energy is:

$$dW_m = \frac{\partial W_m}{\partial \lambda} d\lambda + \frac{\partial W_m}{\partial x} dx$$

同時：
and it is also:

$$dW_m = i d\lambda - f^e dx$$

兩個導數相等，直接代入，得：
So that we can make a direct equivalence between the derivatives and:

$$f^e = -\frac{\partial W_m}{\partial x}$$

相對於直線運動，旋轉運動的轉矩 T^e 取代力 f^e，角位移[31] θ 取代線性位移[32] x。轉矩和角度的乘積與力和距離的乘積單位相同（兩者的單位都是功，國際單位為牛頓—米[33]或焦耳[34]）。
In the case of rotary, as opposed to linear, motion, torque T^e takes the place of force f^e and angular displacement θ takes the place of linear displacement x. Note that the product of torque and angle has the same units as the product of force and distance (both have

[31] 角位移 angular displacement
[32] 線性位移 linear displacement
[33] 牛頓—米 Newton-meter
[34] 焦耳 Joule

units of work, which in the International System of units is Newton-meters or Joules.

很多情況下，在電的線性範圍內，電感[35]是機器位置 x 的函數
In many cases we might consider a system which is electrically linear, in which case inductance is a function only of the mechanical position x.

$$\lambda(x) = L(x)i$$

本例中，假設能量積分由$\lambda=0$ 開始（對 x 積分的部分為 0），
In this case, assuming that the energy integral is carried out from $\lambda = 0$ (so that the part of the integral carried out over x is zero),

$$W_m = \int_0^{\lambda} \frac{1}{L(x)} \lambda d\lambda = \frac{1}{2} \frac{\lambda^2}{L(x)}$$

使得
This makes

$$f^e = -\frac{1}{2}\lambda^2 \frac{\partial}{\partial x} \frac{1}{L(x)}$$

其數值相等於：
Note that this is numerically equivalent to

$$f^e = -\frac{1}{2}i^2 \frac{\partial}{\partial x} L(x)$$

這些關係只有在線性系統才成立。太早將 $L(x)i = \lambda$ 代入公式，會產生誤差：若是在線性系統，會有符號的誤差；而在非線性系統，其結果根本就是錯誤。
This is true only in the case of a linear system. Note that substituting $L(x)i = \lambda$ too early in the derivation produces erroneous results: in the case of a linear system it produces a sign error, but in the case of a nonlinear system it is just wrong.

3.1.1 例：簡單電磁線圈

3.1.1 Example: simple solenoid
圖 5 所示的磁驅動器[36]，包括一件鐵磁性（磁導非常高）材料做成的圓棒，可以在固定元件裡做軸向(x-方向)移動，固定元件也以高導磁性材料做成，一個 N 匝的線圈通以電流 I。圓棒的半徑為 R，距離固定元件的底面為可變尺寸 x。此圓棒另一端在徑向與固定元件之間有氣隙 g，假設 $g<<R$。若此氣隙的軸向長度為 $\ell = R/2$，則兩端的氣隙面積相同。
Consider the magnetic actuator shown in cartoon form in Figure 5. The actuator consists of a circular rod of ferromagnetic material (very highly permeable) that can move axially (the x-direction) inside of a stationary piece, also made of highly permeable material. A coil of N turns carries a current I. The rod has a radius R and spacing from the flat end of the stator is the variable dimension x. At the other end there is a radial clearance between

[35] 電感 inductance
[36] 驅動器 actuator

the rod and the stator g. Assume $g<<R$. If the axial length of the radial gaps is ℓ = $R/2$, the area of the radial gaps is the same as the area of the gap between the rod and the stator at the variable gap.

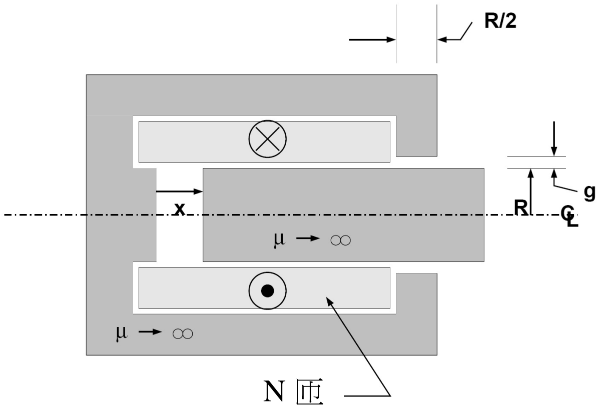

圖 5：磁驅動器

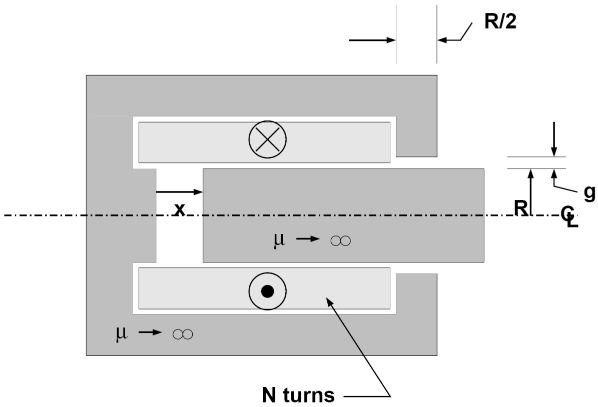

Figure 5: Solenoid Actuator

可變氣隙的磁導為：
The permeances of the variable width gap is :

$$\mathcal{P}_1 = \frac{\mu_0 \pi R^2}{x}$$

若氣隙尺寸比圓棒的半徑為小，另一端的徑向氣隙之磁導為：
and the permeance of the radial clearance gap is, if the gap dimension is small compared with the radius:

$$\mathcal{P}_2 = \frac{2\mu_0 \pi R\ell}{g} = \frac{\mu_0 \pi R^2}{g}$$

線圈[37]的電感為：
The inductance of the coil system is:

[37] 線圈 coil

$$L = \frac{N^2}{\mathcal{R}_1 + \mathcal{R}_2} = N^2 \frac{\mathcal{P}_1 \mathcal{P}_2}{\mathcal{P}_2 + \mathcal{P}_2} = \frac{\mu_0 \pi R^2 N^2}{x + g}$$

磁場能量：
Magnetic energy is :

$$W_m = \int_0^{\lambda_0} i d\lambda = \frac{1}{2} \frac{\lambda^2}{L(x)} = \frac{\lambda_0^2}{2} \frac{x + g}{\mu_0 \pi R^2 N^2}$$

所以，電產生的力為：
And then, of course, force of electric origin is:

$$f^e = -\frac{\partial W_m}{\partial x} = -\frac{\lambda_0^2}{2} \frac{d}{dx} \frac{1}{L(x)}$$

式中：
Here that is easy to carry out:

$$\frac{d}{dx} \frac{1}{L} = \frac{1}{\mu_0 \pi R^2 N^2}$$

因此，力乃為：
So that the force is:

$$f^e(x) = -\frac{\lambda_0^2}{2} \frac{1}{\mu_0 \pi R^2 N^2}$$

若此系統由一電流激磁[38]，可以代入磁通，得：
Given that the system is to be excited by a current, we may at this point substitute for flux:

$$\lambda = L(x) i = \frac{\mu_0 \pi R^2 N^2 i}{x + g}$$

則總力可視為：
And then total force maybe seen to be:

$$f^e = -\frac{\mu_0 \pi R^2 N^2}{(x + g)^2} \frac{i^2}{2}$$

此處負號表示該力趨向於減小 x，亦即縮小空氣隙。

The force is ‘negative’ in the sense that it tends to reduce x, or to close the gap.

$$\mathcal{P}_1 = \frac{\mu_0 \pi R^2}{x}$$

[38] 激磁 excited

3.1.2 多激磁系統

大部分電機機械都可能具備一個以上的激磁源（多於一個線圈）。此種情況，能量守恆可寫為：

3.1.2 Multiply Excited Systems

There may be (and in most electric machine applications there will be) more than one source of electrical excitation (more than one coil). In such systems we may write the conservation of energy expression as:

$$dW_m = \sum_k i_k d\lambda_k - f^e dx$$

上式表示輸入磁場的能量是從所有線圈（此例為 k 個）輸入的電能和。計算儲存於系統的總能量就需要對所有的線圈積分（請注意這些線圈之間可能，而且通常都會有互感）。

which simply suggests that electrical input to the magnetic field energy storage is the sum (in this case over the index k) of inputs from each of the coils. To find the total energy stored in the system it is necessary to integrate over all of the coils (which may and in general will have mutual inductance).

$$W_m = \int \underline{i} \cdot d\underline{\lambda}$$

當然如果此系統為守恆，$W_m(\lambda_1,\lambda_2,\ldots,x)$即為單一值，積分路徑並不影響積分結果。

Of course, if the system is conservative, $W_m(\lambda_1,\lambda_2,\ldots,x)$ is uniquely specified and so the actual path taken in carrying out this integral will not affect the value of the resulting energy.

3.1.3 輔能

我們常常用電感而不用其倒數來描述系統，從而使用電流會比使用磁通適宜。使用一個稱之為「輔能[39]」的新能量變數，將會很方便：

3.1.3 Coenergy

We often will describe systems in terms of inductance rather than its reciprocal, so that current, rather than flux, appears to be the relevant variable. It is convenient to derive a new energy variable, which we will call co-energy, by:

$$W'_m = \sum_i \lambda_i i_i - W_m$$

此處（單一機器變數）能量的微分即為：

and in this case it is quite easy to show that the energy differential is (for a single

[39] 輔能 co-energy

mechanical variable) simply:

$$dW'_m = \sum_k \lambda_k di_k + f^e dx$$

所產生的力為：
so that force produced is:

$$f_e = \frac{\partial W'_m}{\partial x}$$

3.2 例：同步機

3.2 Example: Synchronous Machine

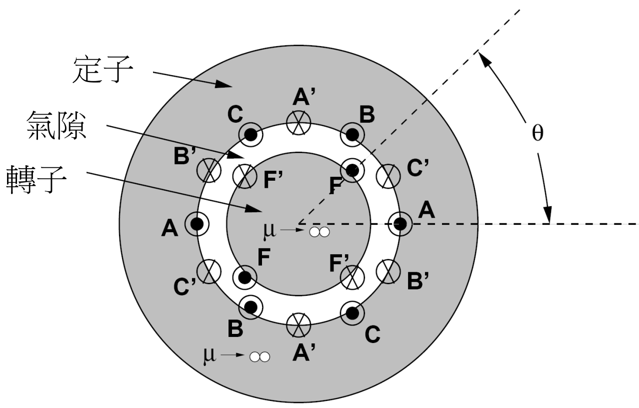

圖 6：同步機示意圖

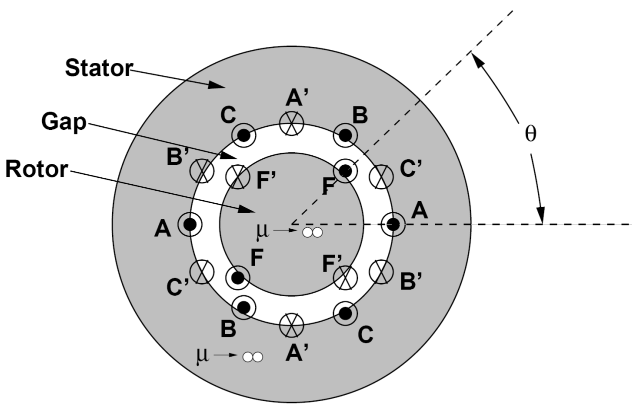

Figure 6: Cartoon of Synchronous Machine

圖 6 所示的簡單電機，在轉子有一個繞組（稱為磁場繞組），而多相[40]的電樞有三個完全相同並均勻分布於周圍的繞組。磁交鏈可以描述為：
Consider a simple electric machine as pictured in Figure 6 in which there is a single winding on a rotor (call it the field winding and a polyphase armature with three identical coils spaced at uniform locations about the periphery. We can describe the flux linkages as:

$$\begin{aligned}
\lambda_a &= L_a i_a + L_{ab} i_b + L_{ab} i_c + M \cos(p\theta) i_f \\
\lambda_b &= L_{ab} i_a + L_a i_b + L_{ab} i_c + M \cos(p\theta - \frac{2\pi}{3}) i_f \\
\lambda_c &= L_{ab} i_a + L_{ab} i_b + L_a i_c + M \cos(p\theta + \frac{2\pi}{3}) i_f \\
\lambda_f &= M \cos(p\theta) i_a + M \cos(p\theta - \frac{2\pi}{3}) i_b + M \cos(p\theta + \frac{2\pi}{3}) + L_f i_f
\end{aligned}$$

假設磁交鏈為轉子位置的正弦函數。這種假設在許多電機機械的各種運轉特性（例如所產生的轉矩的平穩度）都證明是最好的，以致在技術發展方面已將磁交鏈儘量做得極為接近正弦波。往後的各章講義會再討論這些技術，在此只要假設這個關係成立就可以了。另外假設轉子為磁性的「圓形」，亦即定子的自感量和相間互感量不受轉子位置影響。如果每相的繞組都相同(除了角位置之外)，則其自感量相等，如果三相繞組均勻的分布，則相間互感也都相等。
It is assumed that the flux linkages are sinusoidal functions of rotor position. As it turns out, many electrical machines work best (by many criteria such as smoothness of torque production) if this is the case, so that techniques have been developed to make those flux linkages very nearly sinusoidal. We will see some of these techniques in later chapters of these notes. For the moment, we will simply assume these dependencies. In addition, we assume that the rotor is magnetically ’round’, which means the stator self inductances and the stator phase to phase mutual inductances are not functions of rotor position. Note that if the phase windings are identical (except for their angular position), they will have identical self inductances. If there are three uniformly spaced windings the phase-phase mutual inductances will all be the same.

此系統可以簡單的用輔能來描述。由於是多重激磁，取輔能積分時必須很小心（確保在多維空間內循著有效的路徑）。本例實際上有五維，但只有四維是重要的，因為我們可以在所有電流為 0 時定轉子的位置，因而轉子的初始位置對於輔能沒有貢獻。如果轉子在某一角度θ，而四個電流值分別為 i_{a0}、i_{b0}、i_{c0} 及 i_{f0}，正確的積分路徑之一為：
Now, this system can be simply described in terms of coenergy. With multiple excitation it is important to exercise some care in taking the coenergy integral (to ensure that it is taken over a valid path in the multi-dimensional space). In our case there are actually five dimensions, but only four are important since we can position the rotor with all currents at zero so there is no contribution to coenergy from setting rotor position. Suppose the rotor is at some angle θ and that the four currents have values i_{a0}, i_{b0}, i_{c0} and i_{f0}. One of many correct path integrals to take would be:

[40] 多相 polyphase

$$
\begin{aligned}
W'_m &= \int_0^{i_{a0}} L_a i_a di_a \\
&+ \int_0^{i_{b0}} \left(L_{ab} i_{a0} + L_a i_b\right) di_b \\
&+ \int_0^{i_{c0}} \left(L_{ab} i_{a0} + L_{ab} i_{b0} + L_a i_c\right) di_c \\
&+ \int_0^{i_{f0}} \left(M\cos(p\theta) i_{a0} + M\cos(p\theta - \frac{2\pi}{3}) i_{b0} + M\cos(p\theta + \frac{2\pi}{3}) i_{c0} + L_f i_f\right) di_f
\end{aligned}
$$

注意此處第一個積分式是對 i_a 積分而 $i_b = i_c = i_f = 0$，第二個積分式對 i_b 積分而使 $i_a = i_{a0}$ 且 $i_c = i_f = 0$，其餘類推。

結果：

Note here that the first integral is taken over i_a with $i_b = i_c = i_f = 0$, and the second integral is over i_b with $i_a = i_{a0}$ and $i_c = i_f = 0$, etc.
The result is:

$$
\begin{aligned}
W'_m &= \frac{1}{2} L_a \left(i_{a0}^2 + i_{b0}^2 + i_{co}^2\right) + L_{ab}\left(i_{ao} i_{b0} + i_{ao} i_{c0} + i_{co} i_{b0}\right) \\
&+ M i_{f0} \left(i_{a0}\cos(p\theta) + i_{b0}\cos(p\theta - \frac{2\pi}{3}) + i_{c0}\cos(p\theta + \frac{2\pi}{3})\right) + \frac{1}{2} L_f i_{f0}^2
\end{aligned}
$$

既然轉子位置 θ 不會影響定子的電感量，轉矩就很容易計算：

Since there are no variations of the stator inductances with rotor position θ, torque is easily given by:

$$
T_e = \frac{\partial W'_m}{\partial \theta} = -pM i_{f0}\left(i_{a0}\sin(p\theta) + i_{b0}\sin(p\theta - \frac{2\pi}{3}) + i_{co}\sin(p\theta + \frac{2\pi}{3})\right)
$$

3.2.1 電流驅動的同步機

假設可以使用下列電流來驅動同步機：

3.2.1 Current Driven Synchronous Machine

Now assume that we can drive this thing with currents:

$$
\begin{aligned}
i_{a0} &= I_a \cos \omega t \\
i_{b0} &= I_a \cos \omega \left(t - \frac{2\pi}{3} \right) \\
i_{c0} &= I_a \cos \omega \left(t + \frac{2\pi}{3} \right) \\
i_{f0} &= I_f
\end{aligned}
$$

(譯註：上列第 2、3 兩式的左括弧位置似有錯誤，認為應該如下：)

$$
\begin{aligned}
i_{a0} &= I_a \cos \omega t \\
i_{b0} &= I_a \cos\left(\omega t - \frac{2\pi}{3} \right) \\
i_{c0} &= I_a \cos\left(\omega t + \frac{2\pi}{3} \right) \\
i_f &= I_f
\end{aligned}
$$

並且假設轉子以同步速轉動：
and assume the rotor is turning at synchronous speed:

$$p\theta = \omega t + \delta_i$$

因$\cos x \sin y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$，上述的轉矩可以化為：
Noting that *cos x*sin y = 1/2sin(x - y) + 1/2sin(x + y)*, we find the torque expression above to be:

$$
\begin{aligned}
T_e \;\; = -pMI_aI_f \quad & \left(\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin\left(2\omega t + \delta_i\right) \right) \\
& + \left(\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin\left(2\omega t + \delta_i - \frac{4\pi}{3}\right) \right) \\
& + \left(\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin\left(2\omega t + \delta_i + \frac{4\pi}{3}\right) \right)
\end{aligned}
$$

(譯註：上式似乎掉了一對括弧，認為應該如下：)

$$
\begin{aligned}
T_e = -pMI_aI_f & \left\{ \left[\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin(2\omega t + \delta_i) \right] \right. \\
& + \left[\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin\left(2\omega t + \delta_i - \frac{4\pi}{3}\right) \right] \\
& \left. + \left[\frac{1}{2}\sin\delta_i + \frac{1}{2}\sin\left(2\omega t + \delta_i + \frac{4\pi}{3}\right) \right] \right\}
\end{aligned}
$$

靠左邊的三個正弦(sine)函數相加，而靠右邊的三個互相抵銷，結果：

The sine functions on the left add and the ones on the right cancel, leaving:

$$T_e = -\frac{3}{2} p M I_a I_f \sin \delta_i$$

這是說明同步機的一個方法，如果轉子的速率和電流的頻率一致，則所產生的轉矩為穩定值。轉矩和電流轉矩角 δ_i 有關。事實上，此種機器一般並非在電流源之下運轉，以後我們將談到實際的運轉。

And this is indeed one way of looking at a synchronous machine, which produces steady torque if the rotor speed and currents all agree on frequency. Torque is related to the current torque angle δ_i. As it turns out such machines are not generally run against current sources, but we will take up actual operation of such machines later.

$$\begin{aligned}
i_{a0} &= I_a \cos \omega t \\
i_{b0} &= I_a \cos \omega \left(t - \frac{2\pi}{3} \right) \\
i_{c0} &= I_a \cos \omega \left(t + \frac{2\pi}{3} \right) \\
i_{f0} &= I_f
\end{aligned}$$

$$p\theta = \omega t + \delta_i$$

4. 磁場的說明:連續的介質

前面所介紹的利用集合參數方法和虛擬功原理[41]雖然可以解釋基本機電設備，但電機機械有很多現象必須用連續介質法進一步詳細解析。本節將以坡印庭定理[42]來說明以磁場為基礎的能量流，然後以馬克斯威爾應力張量解釋以磁場為基礎的力。這兩種方法都有助於進一步解釋電機機械的各種現象。

4 Field Descriptions: Continuous Media

While a basic understanding of electromechanical devices is possible using the lumped parameter approach and the principle of virtual work as described in the previous section, many phenomena in electric machines require a more detailed understanding which is afforded by a continuum approach. In this section we consider a fields-based approach to energy flow using Poynting's Theorem and then a fields based description of forces using the Maxwell Stress Tensor. These techniques will both be useful in further analysis of what happens in electric machines.

4.1 以磁場描述能量流：坡印庭定理

從法拉第定律[43]開始：

---

[41] 虛擬功原理 principle of virtual work
[42] 坡印庭定理 Poynting's Theorem
[43] 法拉第定律 Faraday's Law

4.1 Field Description of Energy Flow: Poyting's Theorem

Start with Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

安培定律[44]：
and Ampere's Law:

$$\nabla \times \vec{H} = \vec{J}$$

第一式乘以 $\vec{H}$ 而第二式乘以 $\vec{E}$ ，然後兩式相減，得：
Multiplying the first of these by $\vec{H}$ and the second by $\vec{E}$ and taking the difference:

$$\vec{H} \cdot \nabla \times \vec{E} - \vec{E} \cdot \nabla \times \vec{H} = \nabla \cdot \left(\vec{E} \times \vec{H}\right) = -\vec{H} \cdot \frac{\partial \vec{B}}{dt} - \vec{E} \cdot \vec{J}$$

上式等號左邊是電磁能量流的散度[45]：
On the left of this expression is the divergence of electromagnetic energy flow:

$$\vec{S} = \vec{E} \times \vec{H}$$

此處 $\vec{S}$ 即為著名的坡印亭流[46]，也就是電磁場內的功率（其國際單位為瓦／平方公尺。）右邊有兩項：$\vec{H} \cdot \frac{\partial \vec{B}}{dt}$ 是磁場內儲存的能量變化率；第二項 $\vec{E} \cdot \vec{J}$ 看起來很像損失的功率。每一項都要再詳細討論，此刻只要注意向量計算的發散定理會產生：
Here, $\vec{S}$ is the celebrated Poynting flow which describes power in an electromagnetic field sysstem. (The units of this quantity is watts per square meter in the International System). On the right hand side are two terms: $\vec{H} \cdot \frac{\partial \vec{B}}{dt}$ is rate of change of magnetic stored energy. The second term, $\vec{E} \cdot \vec{J}$ looks a lot like power dissipation. We will discuss each of these in more detail. For the moment, however, note that the divergence theorem of vector calculus yields:

$$\int_{\text{volume}} \nabla \cdot \vec{S} dv = \oiint \vec{S} \cdot \vec{n} da$$

亦即坡印亭能量流的散度體積分和在該體積表面的坡印亭能量流相等。此積分式可以寫為：
that is, the volume integral of the divergence of the Poynting energy flow is the same as the Poynting energy flow over the surface of the volume in question. This integral becomes:

[44] 安培定律 Ampere's Law
[45] 散度 divergence
[46] 坡印亭流 Poynting flow

$$\oiint \vec{S}\cdot\vec{n}da = -\int_{\text{volume}}\left(\vec{E}\cdot\vec{J}+\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right)dv$$

它可以解釋為流進某一空間範圍的總能量，等於該範圍內儲存能量的變化率加上看似損失的項對該範圍的體積分。在結束本節討論之前，請注意如果系統內有任何物質的移動，可以使用經驗公式在相對運動的兩邊互相轉換電場。下一式中，加撇號的框架相對於不加撇號的框架以 $\vec{v}$ 速度移動：

which is simply a realization that the total energy flow into a region of space is the same as the volume integral over that region of the rate of change of energy stored plus the term that looks like dissipation. Before we close this, note that, if there is motion of any material within the system, we can use the empirical expression for transformation of electric field between observers moving with respect to each other. Here the 'primed' frame is moving with respect to the 'unprimed' frame with the velocity $\vec{v}$

$$\vec{E}' = \vec{E} + \vec{v}\times\vec{B}$$

此一轉換公式說明一個移動的帶電體，例如電子，在電場和磁場內所受的影響，假如有物質在所觀察的系統內移動，而其速度為 $\vec{v}$，以致在沒有移動物質的框架內（也就是物質本身的框架）可以測量到 $\vec{E}'$ ，則電場和電流密度的乘積為：

This transformation describes, for example, the motion of a charged particle such as an electron under the influence of both electric and magnetic fields. Now, if we assume that there is material motion in the system we are observing and if we assign $\vec{v}$ to be the velocity of that material, so that $\vec{E}'$ measured in a frame in which thre is no material motion (that is the frame of the material itself), the product of electric field and current density becomes:

$$\vec{E}\cdot\vec{J} = \left(\vec{E}'-\vec{v}\times\vec{B}\right)\cdot\vec{J} = \vec{E}'\cdot\vec{J} - \left(\vec{v}\times\vec{B}\right)\cdot\vec{J} = \vec{E}'\cdot\vec{J} + \vec{v}\cdot\left(\vec{J}\times\vec{B}\right)$$

上式中我們用到純量三重積的定理：純量[47]相乘（點相乘）與向量[48]相乘（叉積）的順序可以互換；而將向量相乘（叉積）項的順序反過來會改變其乘積的符號。上列計算的解釋為：

In the last step we used the fact that in a scalar triple product the order of the scalar (dot) and vector (cross) products can be interchanged and that reversing the order of terms in a vector (cross) product simply changes the sign of that product. Now we have a ready interpretation for what we have calculated:

如果加撇號的座標系統[49]是移動物質的框架，

If the 'primed' coordinate system is actually the frame of material motion,

$$\vec{E}'\cdot\vec{J} = \frac{1}{\sigma}|\vec{J}|^2$$

[47] 純量 scalar

[48] 向量 vector

[49] 座標系統 coordinate system

它可以單純的被視為損失，如果物質的導電係數σ為正，其值定為正。最後一項明顯的就是從電磁到機械型態的能量轉換：
which is easily seen to be dissipation and is positive definite if material conductivity σ is positive. The last term is obviously conversion of energy from electromagnetic to mechanical form:

$$\vec{v} \cdot \left(\vec{J} \times \vec{B}\right) = \vec{v} \cdot \vec{F}$$

式中我們得出力的密度，為：
where we have now identified force density to be:

$$\vec{F} = \vec{J} \times \vec{B}$$

此即羅倫茲力定律[50]，該定律說明電流與磁場互相作用以產生力。但它並非機電系統內產生的所有力。我們前面已經談過，改變儲存磁能的機構幾何形狀也會產生力。幸好只要使用磁場就可以充分描述機電的力，這將是下一節要談的主題。
This is the Lorentz Force Law, which describes the interaction of current with magnetic field to produce force. It is not, however, the complete story of force production in electromechanical systems. As we learned earlier, changes in geometry which affect magnetic stored energy can also produce force. Fortunately, a complete description of electromechanical force is possible using only magnetic fields and that is the topic of our next section.

4.2 以磁場描述力：馬克斯威爾應力張量

電磁產生的力是從電場和磁場轉換而得，是這些場的效果，因此只要知道這些場的大小，就可以計算出來。事實上，如果某一面積能將一物體完全包圍，則作用於該物體的力即為力的密度或稱為牽引力者對該面積的積分。

4.2 Field Description of Forces: Maxwell Stress Tensor

Forces of electromagnetic origin, because they are transferred by electric and magnetic fields, are the result of those fields and may be calculated once the fields are known. In fact, if a surface can be established that fully encases a material body, the force on that body can be shown to be the integral of force density, or traction over that surface.

一部機器（見上）的氣隙表面電流密度和磁通密度的叉積所算得的牽引力 τ ，因為從實證所得的 Lorentz 力定律而顯得有意義：只要知道電流密度（向量）和磁通密度（向量），在許多種電機機械內就足以描述力。但因為電機機械含有導磁性材料，即使沒有可見的電流，磁場也對導磁性材料產生力，因此我們需要了解這些材料是怎麼產生力。下式是力密度的經驗公式：
The traction τ derived by taking the cross product of surface current density and flux density on the air-gap surface of a machine (above) actually makes sense in view of the empirically derived Lorentz Force Law: Given a (vector) current density and a (vector) flux density. This is actually enough to describe the forces we see in many machines, but

[50] 羅倫茲力定律 Lorentz Force Law

since electric machines have permeable magnetic material and since magnetic fields produce forces on permeable material even in the absence of macroscopic currents it is necessary to observe how force appears on such material. A suitable empirical expression for force density is:

$$\vec{F} = \vec{J} \times \vec{B} - \frac{1}{2}\left(\vec{H} \cdot \vec{H}\right)\nabla\mu$$

式中 $\vec{H}$ 是磁場強度，μ 是導磁係數。
請注意電流密度是磁場強度的旋度[51]，所以：
Where $\vec{H}$ is the magnetic field intensity and μ is the permeability.
Now, note that current density is the curl of magnetic field intensity, so that:

$$\begin{aligned}\vec{F} &= \left(\nabla \times \vec{H}\right) \times \mu\vec{H} - \frac{1}{2}\left(\vec{H} \cdot \vec{H}\right)\nabla\mu \\ &= \mu\left(\nabla \times \vec{H}\right) \times \vec{H} - \frac{1}{2}\left(\vec{H} \cdot \vec{H}\right)\nabla\mu\end{aligned}$$

因：
And, since:

$$\left(\nabla \times \vec{H}\right) \times \vec{H} = \left(\vec{H} \cdot \nabla\right)\vec{H} - \frac{1}{2}\nabla\left(\vec{H} \cdot \vec{H}\right)$$

力密度乃為
Force density is:

$$\begin{aligned}\vec{F} &= \mu\left(\vec{H} \cdot \nabla\right)\vec{H} - \frac{1}{2}\mu\nabla\left(\vec{H} \cdot \vec{H}\right) - \frac{1}{2}\left(\vec{H} \cdot \vec{H}\right)\nabla\mu \\ &= \mu\left(\vec{H} \cdot \nabla\right)\vec{H} - \nabla\left(\frac{1}{2}\mu\left(\vec{H} \cdot \vec{H}\right)\right)\end{aligned}$$

此式可用其分量寫成，力的第 i 維分量為：
This expression can be written by components: the component of force in the i'th dimension is:

$$F_i = \mu\sum_k\left(H_k\frac{\partial}{\partial x_k}\right)H_i - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\mu\sum_k H_k^2\right)$$

上式第一項可寫成：
The first term can be written as:

$$\mu\sum_k\left(H_k\frac{\partial}{\partial x_k}\right)H_i = \sum_k\frac{\partial}{\partial x_k}\mu H_k H_i - H_i\sum_k\frac{\partial}{\partial x_k}\mu H_k$$

最末項則為磁通密度的散度，其值為 0：
The last term in this expression is easily shown to be divergence of magnetic flux density, which is zero:

---

51 旋度 curl

$$\nabla \cdot \vec{B} = \sum_k \frac{\partial}{\partial x_k} \mu H_k = 0$$

將這些代入後，力的密度可以簡化為：

Using this, we can write force density in a more compact form as:

$$F_k = \frac{\partial}{\partial x_i} \left(\mu H_i H_k - \frac{\mu}{2} \delta_{ik} \sum_n H_n^2 \right)$$

式中我們用了 Kroneker δ，若 $i=k$ 則 $\delta_{ik} = 1$，否則 $\delta_{ik} = 0$。

請注意力的密度是張量的散度的型態：

where we have used the Kroneker delta $\delta_{ik} = 1$ if $i = k$, 0 otherwise.

Note that this force density is in the form of the divergence of a tensor:

$$F_k = \frac{\partial}{\partial x_i} T_{ik}$$

也可寫成

or

$$\vec{F} = \nabla \cdot \underline{\underline{T}}$$

在此情況，若是某物體能被閉合的表面包圍，則作用於此物體的力可以用散度定理來計算：

In this case, force on some object that can be surrounded by a closed surface can be found by using the divergence theorem:

$$\vec{f} = \int_{\mathrm{vol}} \vec{F} dv = \int_{\mathrm{vol}} \nabla \cdot \underline{\underline{T}} dv = \oiint \underline{\underline{T}} \cdot \vec{n} da$$

或者，若是表面的牽引力為 $\tau_i = \sum_k T_{ik} n_k$，式中 n 是表面的正交向量，則 i 方向的總力為：

or, if we note surface traction to be $\tau_i = \sum_k T_{ik} n_k$, where n is the surface normal vector, then the total force in direction i is just:

$$\vec{f} = \oint_s \tau_i da = \oint \sum_k T_{ik} n_k da$$

解釋這些現象其實不像公式那麼複雜。磁場描述的力給出表面牽引力（就是表面單位面積的力）的簡單概念。將牽引力對某物體面積積分，就可以得到物體全部的力。請注意，本公式中的下標如果像此處這樣重複，有時會省略積分符號，寫成 $\tau_i = \sum_k T_{ik} n_k = T_{ik} n_k$

The interpretation of all of this is less difficult than the notation suggests. This field description of forces gives us a simple picture of surface traction, the force per unit area on a surface. If we just integrate this traction over the area of some body we get the whole force on the body.

Note one more thing about this notation. Sometimes when subscripts are repeated as they are here the summation symbol is omitted. Thus we would write $\tau_i = \sum_k T_{ik} n_k = T_{ik} n_k$.

4.3 例：線性感應機

圖 7 是非常簡化的單邊線性感應馬達，實際的線性感應機並非這個樣子，但是經由對稱論證，我們在此進行的分析，可以應用到這一類的其他機器。

4.3 Example: Linear Induction Machine

Figure 7 shows a highly simplified picture of a single sided linear induction motor. This is not how most linear induction machines are actually built, but it is possible to show through symmetry arguments that the analysis we can carry out here is actually valid for other machines of this class.

此機器包括一個定子(上表面)，以高導磁區的表面電流示意之，移動元件則為高導磁區表面一層薄的導電材料。移動元件(或稱之為「梭子[52]」)相對於定子的速率為 *u*，在 *x* 方向移動。定子表面電流密度假設為：

This machine consists of a stator (the upper surface) which is represented as a surface current on the surface of a highly permeable region. The moving element consists of a thin layer of conducting material on the surface of a highly permeable region. The moving element (or 'shuttle') has a velocity *u* with respect to the stator and that motion is in the *x* direction. The stator surface current density is assumed to be:

$$K_z = \mathrm{Re}\left\{\underline{K}_z e^{j(\omega t - kx)}\right\}$$

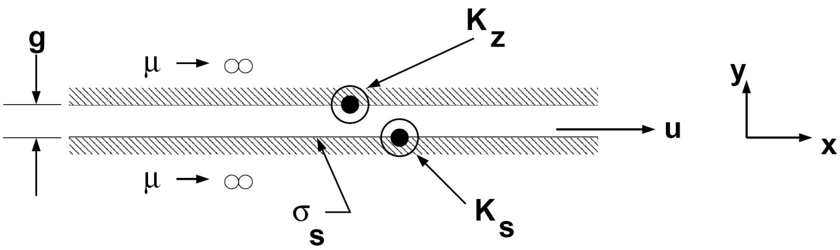

圖 7：單邊線性感應機之簡單模型

Figure 7: Simple model of single sided linear induction machine

此處我們忽略一些重要效應，例如定子和梭子的有限長度造成的影響。雖然這些效應相當重要，但我們將留待後面再討論，因為這些效應是線性馬達很有趣的題目。

Note that we are ignoring some important effects, such as those arising from finite length of the stator and of the shuttle. Such effects can be quite important, but we will leave those until later, as they are what make linear motors interesting.

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[52] 梭子 shuttle

從梭子方面觀察，在移動方向的尺度為 *x'- x - ut'*，相對的頻率為：
Viewed from the shuttle for which the dimension in the direction of motion is *x' - x – ut'*, the relative frequency is:

$$\omega t - kx = (\omega - ku)\, t - kx' = \omega_s t - kx'$$

由於梭子表面可以流通電流，且該電流乃經由定子電流激起的磁場而感應產生，我們可以合理的假設梭子的電流和定子電流有相同的形式，即：
Now, since the shuttle surface can support a surface current and is excited by magnetic fields which are in turn excited by the stator currents, it is reasonable to assume that the form of rotor current is the same as that of the stator:

$$K_s = \mathrm{Re}\left\{\underline{K}_s e^{j(\omega_s t - kx')}\right\}$$

在此情況，安培定律為：
Ampere's Law is, in this situation:

$$g\frac{\partial H_y}{\partial x} = K_z + K_s$$

其複值幅[53]為：
which is, in complex amplitudes:

$$\underline{H}_y = \frac{\underline{K}_z + \underline{K}_s}{-jkg}$$

假設此處 *z*-方向為均勻，法拉第定律的 *y*-方向分量為：
The *y*- component of Faraday's Law is, assuming the problem is uniform in the *z*-direction:

$$-j\omega_s \underline{B}_y = jk\underline{E}'_z$$

即：
or

$$\underline{E}'_z = -\frac{\omega_s}{k}\mu_0 \underline{H}_y$$

經簡單代數運算，可以得到轉子表面電流和氣隙磁場的複數幅值：
A bit of algebraic manipulation yields expressions for the complex amplitudes of rotor surface current and gap magnetic field:

$$\begin{aligned}
\underline{K}_s &= \frac{-j\frac{\mu_0\omega_s\sigma_s}{k^2 g}}{1 + j\frac{mu_0\omega_s\sigma_s}{k^2 g}}\underline{K}_z \\
\underline{H}_y &= \frac{j}{kg}\frac{\underline{K}_z}{1 + j\frac{mu_0\omega_s\sigma_s}{k^2 g}}
\end{aligned}$$

欲解表面牽引力，利用馬克斯威爾應力張量來分析定子底下緊鄰的層面（此層面上

[53] 複值幅 complex amplitude

x-方向的磁場即為 $\underline{H}_x = \underline{K}_z$ 。）所以牽引力為：

To find surface traction, the Maxwell Stress Tensor can be evaluated at a surface just below the stator (on this surface the x-directed magnetic field is simply $\underline{H}_x = \underline{K}_z$. Thus the traction is

$$\tau_x = T_{xy} = \mu_0 H_x H_y$$

其平均值為：

And the average of this is:

$$< \tau_x >= \frac{\mu_0}{2} \text{Re} \left\{ \underline{H}_x \underline{H}_y * \right\}$$

即：

This is:

$$< \tau_x >= \frac{\mu_0}{2} \frac{1}{kg} \frac{|\underline{K}_z|^2 \frac{\mu_0 \omega_s \sigma_s}{k^2 g}}{1 + \left(\frac{\mu_0 \omega_s \sigma_s}{k^2 g} \right)^2}$$

依坡印庭定理，在 y-方向的電磁功率為：

Now, if we consider electromagnetic power flow (Poynting's Theorem): in the y-direction:

$$S_y = E_z H_x$$

因為在梭子的框架 $\underline{E}_z' = -\frac{\omega_s}{k} \mu_0 \underline{H}_y$

And since in the frame of the shuttle $\underline{E}_z' = -\frac{\omega_s}{k} \mu_0 \underline{H}_y$

$$< S_y' >= -\frac{1}{2} \frac{\omega_s}{k} \frac{\mu_0}{kg} \frac{\frac{\mu_0 \omega_s \sigma_s}{k^2 g}}{1 + \left(\frac{\mu_0 \omega_s \sigma_s}{k^2 g} \right)^2} |\underline{K}_z|^2 = -\frac{\omega_s}{k} < \tau_x >$$

相同的，在定子的框架內計算：

Similarly, evaluated in the frame of the stator:

$$< S_y >= -\frac{\omega}{k} < \tau - x >$$

此乃證明：定子的電磁功率等於梭子的力密度乘以波速。流入梭子的電磁功率等於相同的力密度乘以「差速率」。此兩者之差為轉成機械形式之功，等於力密度乘以梭子速率。

This shows what we already suspected: the electromagnetic power flow from the stator is the force density on the shuttle times the wave velocity. The electromagnetic ower flow into the shuttle is the same force density times the 'slip' velocity. The difference between these two is the power converted to mechanical form and it is the force density times the shuttle velocity.

4.4 旋轉機械

此公式用在旋轉機械[54]有些奇怪，因為如果要將力做積分以得到總力，方向向量必須有常數單位元素。當然，在圓柱座標，方向向量不用常數單位元素。但仔細了解牽引力的方向以及如何積分之後，我們可以把馬克斯威爾應力張量應用在旋轉電機。

4.4 Rotating Machines

The use of this formulation in rotating machines is a bit tricky because, at lest formally, directional vectors must have constant identity if an integral of forces is to become a total force. In cylindrical coordinates, of course, the directional vectors are not of constant identity. However, with care and understanding of the direction of traction and how it is integrated we can make use of the MST approach in rotating electric machines.

讓我們回頭複習圓柱轉子的轉矩，計算圓週的力時，我們了解圓柱的法線向量就是其徑向的單位向量，所以圓週的牽引力為：
Now, if we go back to the case of a circular cylinder and are interested in torque, it is pretty clear that we can compute the circumferential force by noting that the normal vector to the cylinder is just the radial unit vector, and then the circumferential traction must simply be:

$$\tau_\theta = \mu_0 H_r H_\theta$$

假設轉子內部沒有磁通，單純的對表面積分將得出角向的力。這相當於轉子表面環繞著連續的無限多小方格，一個面剛好在轉子之外，其法線朝外；另一個面剛好在內，其法線朝內（當然此內層面的馬克斯威爾應力張量為 0）。然後乘以半徑（力臂）得出轉矩。最後請注意，如果轉子使用高導磁材料做成，緊鄰轉子外面的角向磁場等於表面電流密度。
Assuming that there are no fluxes inside the surface of the rotor, simply integrating this over the surface gives azimuthal force. In principal this is the same as surrounding the surface of the rotor by a continuum of infinitely small boxes, one surface just outside the rotor and with a normal facing outward, the other surface just inside with normal facing inward. (Of course the MST is zero on this inner surface). Then multiplying by radius (moment arm) gives torque. The last step is to note that, if the rotor is made of highly permeable material, the azimuthal magnetic field just outside the rotor is equal to surface current density.

5. 連續介質的一般公式

一個系統不只存在多條電流路徑，而是連續並且均勻分佈的電流路徑時，輔能可以表示成：

5 Generalization to Continuous Media

Now, consider a system with not just a multiplicity of circuits but a continuum of

[54] 旋轉機械 rotating machine

current-carrying paths. In that case we could identify the co-energy as:

$$W_m' = \int_{\text{area}} \int \lambda(\vec{a}) d\vec{J} \cdot d\vec{a}$$

式中的面積與所有電流導體相交。由於電流的散度為 0，此面積可以選為和每一條「細電流絲[55]」垂直。磁通λ則循著每一細電流絲路徑計算（由於電流散度為 0，所以存在此路徑)。則磁通為：

where that area is chosen to cut all of the current carrying conductors. This area can be picked to be perpedicular to each of the current filaments since the divergence of current is zero. The flux λ is calculated over a path that coincides with each current filament (such paths exist since current has zero divergence). Then the flux is:

$$\lambda(\vec{a}) = \int \vec{B} \cdot d\vec{n}$$

若使用向量勢 $\vec{A}$ ，且磁通密度為：

Now, if we use the vector potential $\vec{A}$ for which the magnetic flux density is:

$$\vec{B} = \nabla \times \vec{A}$$

則與任一細電流絲互相交鏈的磁通為：

the flux linked by any one of the current filaments is:

$$\lambda(\vec{a}) = \oint \vec{A} \cdot d\vec{\ell}$$

式中 $d\vec{\ell}$ 為圍繞細電流絲的路徑。此式直接表示輔能為：

Where $d\vec{\ell}$ is the path around the current filament. This implies directly that the coenergy is:

$$W_m' = \int_{\text{area}} \int_J \oint \vec{A} \cdot d\vec{\ell} d\vec{J} \cdot d\vec{a}$$

可以選 $d\vec{\ell}$ 和 $d\vec{a}$ 同方向，並與細電流絲平行，則：

Now: it is possible to make $d\vec{\ell}$ coincide with $d\vec{a}$ and be parallel to the current filaments, so that:

$$W_m' = \int_{\text{vol}} \vec{A} \cdot d\vec{J} dv$$

5.1 永久磁鐵

永久磁鐵是電機機械越來越重要的元件。具永久磁鐵的系統常常使用比較特別的方

[55] 細電流絲 current filament

法來分析，使用能夠產生相同磁動勢[56]的等效電流來代表磁鐵。

5.1 Permanent Magnets

Permanent magnets are becoming an even more important element in electric machine systems. Often systems with permanent magnets are approached in a relatively ad-hoc way, made equivalent to a current that produces the same MMF as the magnet itself.

永久磁鐵的系統中，磁通密度 $\vec{B}$ 、磁場強度 $\vec{H}$ 以及磁化強度 $\vec{M}$ 之關係為。：

The constitutive relationship for a permanent magnet relates the magnetic flux density $\vec{B}$ to magnetic field $\vec{H}$ and the property of the magnet itself, the magnetization $\vec{M}$.

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

磁化強度的效果相當於有一電流（稱為安培電流[57]），其密度為：
Now, the effect of the magnetization is to act as if there were a current (called an amperian current) with density:

$$\vec{J}^* = \nabla \times \vec{M}$$

此安培電流的「作用」就像一般產生磁通密度的電流。磁輔能為：
Note that this amperian current ‘acts" just like ordinary current in making magnetic flux density. Magnetic co-energy is:

$$W'_m = \int_{\text{vol}} \vec{A} \cdot \nabla \times d\vec{M} dv$$

其次，向量計算的恆等式： $\nabla \cdot (\vec{C} \times \vec{D}) = \vec{D} \cdot (\nabla \times \vec{C}) - \vec{C} \cdot (\nabla \times \vec{D})$

Next, note the vector identity $\nabla \cdot \left(\vec{C} \times \vec{D} \right) = \vec{D} \cdot \left(\nabla \times \vec{C} \right) - \vec{C} \cdot \left(\nabla \times \vec{D} \right)$ Now,

$$W'_m = \int_{\text{vol}} -\nabla \cdot \left(\vec{A} \times d\vec{M} \right) dv + \int_{\text{vol}} \left(\nabla \times \vec{A} \right) \cdot d\vec{M} dv$$

又 $\vec{B} = \nabla \times \vec{A}$

Then, nothing that $\vec{B} = \nabla \times \vec{A}$:

$$W'_m = - \oiint \vec{A} \times d\vec{M} d\vec{s} + \int_{\text{vol}} \vec{B} \cdot d\vec{M} dv$$

這些積分的第一項（閉合面積）若是對磁鐵外的表面做積分，因為 $\vec{M}$ 為 0，將會消失。只有一個永久磁鐵源的系統，其磁輔能為：

[56] 磁動勢 Magnetomotive Force, MMF
[57] 安培電流 amperian current

The first of these integrals (closed surface) vanishes if it is taken over a surface just outside the magnet, where $\vec{M}$ is zero. Thus the magnetic co-energy in a system with only a permanent magnet source is

$$W'_m = \int_{\text{vol}} \vec{B} \cdot d\vec{M} dv$$

增加通電流的線圈到此系統，是用上明顯的方法。
Adding current carrying coils to such a system is done in the obvious way.